Polymorphism

- Outline
  - *Propositional logic*
  - *First-order logic*
  - *Second-order logic*
  - *System F (Second-order lambda calculus, Girard 1972)*

Propositional Logic Syntax

Propositions:

$$
\begin{array}{rcl}
e & ::= & \top \mid \bot \\
  & \mid & P, Q, R, \ldots \\
  & \mid & \neg e \\
  & \mid & e \wedge e \\
  & \mid & e \vee e \\
  & \mid & e \Rightarrow e
\end{array}
$$

Semantics

- The semantics of constants: $\top = 1, \bot = 0$
- The semantics of a propositional letter is its truth value.
- The semantics of a compound proposition is determined by truth tables.

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \wedge B$
0	0	1
0	1	1
1	0	0
1	1	1

A proof (Pierce's Law)

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow A$	$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
0	0	1	0	1
0	1	1	0	1
1	0	0	1	1
1	1	1	1	1

Sequents

- A *sequent* has the form $\Gamma \vdash \Delta$
- Δ is a list of propositions $\alpha_1, \ldots, \alpha_n$
- Γ is a *context* containing a list of propositions $\beta_1, \ldots, \beta_n$
- We can extend valuations to sequents, to get the folowing semantics:
  - A sequent $\beta_1, \ldots, \beta_n \vdash \alpha_1, \ldots, \alpha_n$ is true if some α_i is true whenever $\beta_1, \ldots, \beta_n$ are all true.

Derivations

- There are two kinds of inference rules
  - *Introduction* rules operate on the right of the turnstile
  - *Elimination* rules operate on the left of the turnstile
- The base axiom

$$\frac{}{\Gamma_1, \alpha, \Gamma_2 \vdash \Delta_1, \alpha, \Delta_2}\ \text{axiom}$$

Rule table

$$\frac{}{\Gamma \vdash \Delta_1, \top, \Delta_2} \qquad \frac{}{\Gamma_1, \bot, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta_1, \alpha, \Delta_2 \quad \Gamma \vdash \Delta_1, \beta, \Delta_2}{\Gamma \vdash \Delta_1, \alpha \wedge \beta, \Delta_2} \qquad \frac{\Gamma_1, \alpha, \beta, \Gamma_2 \vdash \Delta}{\Gamma_1, \alpha \wedge \beta, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta_1, \alpha, \beta, \Delta_2}{\Gamma \vdash \Delta_1, \alpha \vee \beta, \Delta_2} \qquad \frac{\Gamma_1, \alpha, \Gamma_2 \vdash \Delta \quad \Gamma_1, \beta, \Gamma_2 \vdash \Delta}{\Gamma_1, \alpha \vee \beta, \Gamma_2 \vdash \Delta}$$

$$\frac{\Gamma, \alpha \vdash \Delta_1, \beta, \Delta_2}{\Gamma \vdash \Delta_1, \alpha \Rightarrow \beta, \Delta_2} \qquad \frac{\Gamma_1, \beta, \Gamma_2 \vdash \Delta \quad \Gamma_1, \Gamma_2 \vdash \alpha, \Delta}{\Gamma_1, \alpha \Rightarrow \beta, \Gamma_2 \vdash \Delta}$$

Proving the law of excluded middle

- Every proposition is either true or false

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,3}{\vdash A, \neg A}\,2}{\vdash A \vee \neg A}\,1$$

Pierce's law

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A}\,4}{\vdash A \Rightarrow B, A}\,3 \qquad \dfrac{}{A \vdash A}\,5}{(A \Rightarrow B) \Rightarrow A \vdash A}\,2}{\vdash ((A \Rightarrow B) \Rightarrow A) \Rightarrow A}\,1$$

Currying

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A,B \vdash A}\,5 \quad \dfrac{}{A,B \vdash B}\,6}{A,B \vdash A \wedge B}\,4 \quad \dfrac{}{C,A,B \vdash C}\,6}{(A \wedge B) \Rightarrow C, A, B \vdash C}\,3}{(A \wedge B) \Rightarrow C \vdash A \Rightarrow B \Rightarrow C}\,2}{\vdash ((A \wedge B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))}\,1$$

Soundness and completeness

Given two sets of propositions:

- $\Gamma = \beta_1, \ldots, \beta_n$
- $\Delta = \alpha_1, \ldots, \alpha_n$

We say $\Gamma \vDash \Delta$ if every valuation $\mathcal{V}$ that makes all β_i true also makes at least on α_j true.

- Valuations are truth tables.
- The $\vDash$ symbol is pronounced "models."

Definitions

Soundness If $\Gamma \vdash \Delta$, then $\Gamma \vDash \Delta$

Completeness If $\Gamma \vDash \Delta$, then $\Gamma \vdash \Delta$

Soundness

- Assume $\Gamma \vdash \Delta$, prove $\Gamma \vDash \Delta$
- Use induction on the length of the proof $\Gamma \vdash \Delta$

Soundness: the axiom rule

- Suppose the last rule of the proof is the **axiom** rule:

$$\frac{}{\Gamma_1, \alpha, \Gamma_2 \vdash \Delta_1, \alpha, \Delta_2}\ axiom$$

- Consider any valuation $\mathcal{V}$ that makes all propositions in Γ true.
- Then $\mathcal{V}$ makes at least one proposition (α) in Δ true.
- So $\Gamma_1, \alpha, \Gamma_2 \vDash \Delta_1, \alpha, \Delta_2$

Soundness: the implies intro rule

- Suppose the last rule of the proof is the **implies intro** rule.

$$\frac{\Gamma, \alpha \vdash \Delta_1, \beta, \Delta_2}{\Gamma \vdash \Delta_1, \alpha \Rightarrow \beta, \Delta_2}\ \text{implies intro}$$

- By (proof) induction, we know $\Gamma, \alpha \vDash \Delta_1, \beta, \Delta_2$.
- So, any valuation that makes Γ, α true also makes some proposition in $\Delta_1, \beta, \Delta_2$ true.

Soundness: implies intro

- Consider any valuation $\mathcal{V}'$ that makes Γ true:
  - If α is false in $\mathcal{V}'$, then $\alpha \Rightarrow \beta$ is true.
  - If α is true in $\mathcal{V}'$, then so is some proposition in $\Delta_1, \beta, \Delta_2$.
- It follows that $\Gamma \vDash \Delta_1, \alpha \Rightarrow \beta, \Delta_2$

Completeness

Any unfinished branch can be extended to a valuation that is a counterexample.

$$\cfrac{\cfrac{\cfrac{\cfrac{A \vdash B, C}{A, \neg B \vdash C}\,4 \qquad \cfrac{}{A, C \vdash C}\,5}{A, \neg B \vee C \vdash \neg A, C}\,3}{A, \neg B \vee C \vdash \neg A \vee C}\,2}{\vdash A \Rightarrow (\neg B \vee C) \Rightarrow (\neg A \vee C)}\,1$$

The counteraxmple is $A = 1, B = 0, C = 0$.

First-order logic

- Add *atomic formulas* $f(a_1, \ldots, a_n)$ of various aritites.
- Add *atomic predicates* $P(a_1, \ldots, a_m)$ is various arities.

$$\begin{array}{rcl} a & ::= & f(a_1, \ldots, a_n) \\ e & ::= & \top \mid \bot \\ & \mid & P(a_1, \ldots, a_m) \\ & \mid & \neg e \\ & \mid & e \wedge e \\ & \mid & e \vee e \\ & \mid & e \Rightarrow e \\ & \mid & \forall v.e \\ & \mid & \exists v.e \end{array}$$

Some examples

- $\forall x.big(x) \Rightarrow heavy(x)$
- $\forall i.(i+1) > i$
- $\forall i.\exists j.j > i$
- $\exists i.\forall j.j \leq i$

Adding rules for FOL

$$\frac{\Gamma \vdash \Delta_1, P(t), \Delta_2}{\Gamma \vdash \Delta_1, \exists v.P(v), \Delta_2} \text{ exists intro}$$

$$\frac{\Gamma \vdash \Delta_1, P(c), \Delta_2 \quad (\text{new } c)}{\Gamma \vdash \Delta_1, \forall v.P(v), \Delta_2} \text{ all intro}$$

$$\frac{\Gamma_1, P(c), \Gamma_2 \vdash \Delta \quad (\text{new } c)}{\Gamma_1, \exists v.P(v), \Gamma_2 \vdash \Delta} \text{ exists elim}$$

$$\frac{\Gamma_1, \forall v.P(v), \Gamma_2, P(t) \vdash \Delta}{\Gamma_1, \forall v.P(v), \Gamma_2 \vdash \Delta} \text{ all elim}$$

A quantifier DeMorgan law

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\varphi(c) \vdash \varphi(c)}\ 6}{\forall x.\varphi(x) \vdash \varphi(c)}\ 5}{\neg\varphi(c), \forall x.\varphi(x) \vdash}\ 4}{\neg\varphi(c) \vdash \neg(\forall x.\varphi(x))}\ 3}{\exists x.\neg\varphi(x) \vdash \neg(\forall x.\varphi(x))}\ 2}{\vdash (\exists x.\neg\varphi(x)) \Rightarrow \neg(\forall x.\varphi(x))}\ 1$$

Another proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi(c) \vdash \varphi(c)}{\varphi(c) \vdash \exists x.\varphi(x)}\,5 \qquad \psi \vdash \psi}{(\exists x.\varphi(x)) \Rightarrow \psi, \varphi(c) \vdash \psi}\,4}{(\exists x.\varphi(x)) \Rightarrow \psi, \varphi(c) \Rightarrow \psi}\,3}{(\exists x.\varphi(x)) \Rightarrow \psi \vdash \forall x.\varphi(x) \Rightarrow \psi}\,2}{\vdash ((\exists x.\varphi(x)) \Rightarrow \psi) \Rightarrow (\forall x.\varphi(x) \Rightarrow \psi)}\,1$$

Soundness and completeness

- Soundness and completeness are similar to propositional logic
  - *The semantics uses "first-order structures"*
    - *Each function symbol is interpreted as a real function*
    - *Each predicate symbol is interpreted as a real predicate*
    - *Quantification is over the set defined by the structure*

Second-order logic

- In FOL, the variables range over the <u>elements</u> of a structure
- Sometimes it is useful to consider all subsets of a structure
  - *"Each bounded non-empty set of reals has a supremum"*
  - *"Each non-empty set of natural numbers has a minimal element"*
- Second-order logic allows quantification over relations

SOL language definition

The language contains:

- individual variables $x_0, x_1, \ldots$
- constants $c_0, c_1, \ldots$ and n-ary function symbols $f_0^n, f_1^n, \ldots$
- n-ary predicate variables $X_0^n, X_1^n, \ldots$
- n-ary predicate symbols $P_0^n, P_1^n, \ldots$
- the usual connectives $\neg, \wedge, \vee, \Rightarrow, \exists, \forall$

Rules for predicate quantification

$$\frac{\Gamma \vdash \Delta_1, \varphi[\sigma/X], \Delta_2 \quad \text{(some predicate } \sigma)}{\Gamma \vdash \Delta_1, \exists X.\varphi, \Delta_2}\ \exists^2\textbf{intro}$$

$$\frac{\Gamma_1, \varphi[C/X], \Gamma_2 \vdash \Delta \quad \text{(new symbol } C)}{\Gamma_1, \exists X.\varphi, \Gamma_2 \vdash \Delta}\ \exists^2\textbf{elim}$$

$$\frac{\Gamma \vdash \Delta_1, \varphi[C/X], \Delta_2 \quad \text{(new symbol } C)}{\Gamma \vdash \Delta_1, \forall X.\varphi, \Delta_2}\ \forall^2\textbf{intro}$$

$$\frac{\Gamma_1, \forall X.\varphi, \Gamma_2, \varphi[\sigma/X] \vdash \Delta \quad \text{(some predicate } \sigma)}{\Gamma_1, \forall X.\varphi, \Gamma_2 \vdash \Delta}\ \forall^2\textbf{elim}$$

Notation

The substitution $\varphi[\sigma/X]$ means replace each occurrence of $X(t_1, \ldots, t_n)$ in φ with $\sigma(t_1, \ldots, t_n)$

Negation is definable

$$\dfrac{\dfrac{\bot \vdash \bot}{\forall X.X \vdash \bot}}{\vdash \bot \Leftrightarrow \forall X.X}$$

Conjunction is definable

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi, \psi \vdash \varphi \quad \dfrac{\cdots}{\varphi, \psi, \psi \Rightarrow C \vdash C}}{\varphi, \psi, \varphi \Rightarrow (\psi \Rightarrow C) \vdash C}}{\varphi, \psi \vdash (\varphi \Rightarrow (\psi \Rightarrow C)) \Rightarrow C}}{\varphi, \psi \vdash \forall X.((\varphi \Rightarrow (\psi \Rightarrow X)) \Rightarrow X)}}{\vdash (\varphi \wedge \psi) \Rightarrow (\forall X.((\varphi \Rightarrow (\psi \Rightarrow X)) \Rightarrow X))}$$

Operator definitions in SOL

All operators can be defined in terms of $\forall$ and $\Rightarrow$

$$\begin{array}{rcl}
\bot & \Leftrightarrow & \forall X.X \\
\varphi \wedge \psi & \Leftrightarrow & \forall X.(\varphi \Rightarrow (\psi \Rightarrow X)) \Rightarrow X \\
\varphi \vee \psi & \Leftrightarrow & \forall X.((\psi \Rightarrow X) \wedge (\psi \Rightarrow X)) \Rightarrow X \\
\exists x.\varphi(x) & \Leftrightarrow & \forall X.(\forall x.\varphi(x) \Rightarrow X) \Rightarrow X \\
\exists X.\varphi(X) & \Leftrightarrow & \forall Y.(\forall X.(\varphi(X) \Rightarrow Y) \Rightarrow Y
\end{array}$$

The polymorphic lambda calculus

e	$::=$	v	variables
	$\mid$	$\lambda x{:}t.e$	abstraction
	$\mid$	$e_1\ e_2$	application
	$\mid$	$\Lambda X.e$	type abstraction
	$\mid$	$e[t]$	type application
t	$::=$	X	type variables
	$\mid$	$t_1 \to t_2$	function types
	$\mid$	$\forall X.t$	type quantification